Darboux's theorem and Euler-like vector fields

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Abstract

This essay was created for the purposes of demonstrating the ability to competently write about mathematics, as required by the Penn State graduate program for the scheduling of a doctoral comprehensive exam. It was complemented by a presentation delivered in a geometry topics course run by Dmitri Burago. As to the mathematical contents, we prove the Darboux theorem by leveraging the linearizability of vector fields of the form: Euler vector field $+$ higher order terms. This method is taken from a talk titled "Some normal form theorems in differential geometry" given by Eckhard Meinrenken in the Penn State GAP Seminar on 20 February, 2018.

1 Darboux's theorem

As any cartographer knows, it is impossible to design a map of the earth's surface which does not distort any distances. This is a manifestation of Gauss's Theorema Egregium. On the other hand, there *do* exist maps of the earth which preserve all areas, for example the cylindrical projection shown in Figure 1. This is a manifestation of Darboux's theorem.

Theorem 1 (Darboux, 1882)**.** *Two symplectic manifolds of the same dimension are locally isomorphic to one another.*

Remark 2. For surfaces, a symplectic form is the same thing as an area form. A smooth, area-preserving bijection between a patch on the 2-sphere and a patch on the Euclidean plane is the same thing as a local isomorphism of their symplectic structures.

Darboux's theorem is frequently interpreted as saying that symplectic geometry has "no local invariants". This distinguishes it from Riemannian geometry, where curvature obstructs isometric identification of small patches on different manifolds.

Definition 3. A *symplectic form* on a smooth manifold M is a (smooth) closed, nondegenerate 2-form ω. A *symplectic manifold* is a manifold equipped with a preferred symplectic form. An *isomorphism* (or symplectomorphism) between two symplectic manifolds is a diffeomorphism which carries the symplectic form of one to the symplectic form of the other,

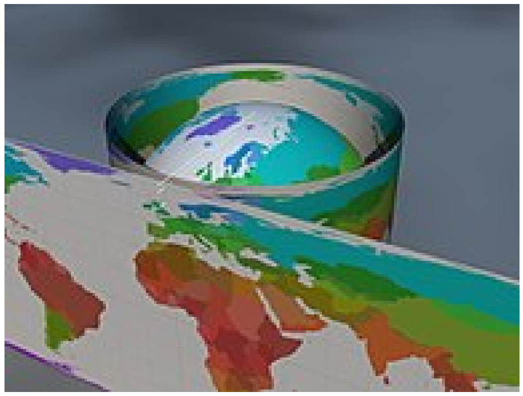

Figure 1: Area-preserving cylindrical projection

Here are three (boring) examples of symplectic forms.

Example 4. The standard area form $\omega = dxdy$ on $\mathbb{R}^2$.

Example 5. The *standard symplectic form* $\omega = \sum_{i=1}^n dx^i dy^i$ on $\mathbb{R}^{2n}$, where the coordinate functions are denoted $x^1, \ldots, x^n, y^1, \ldots, y^n$, effectively the n-fold product of Example 4.

Example 6. Any constant (hence closed), nondegenerate 2-form on Euclidean space.

While, at first glance, it would appear that Example 6 is more general than Example 5, it turns out

Proposition 7. *If ω is a constant, nondegenerate 2-form on $\mathbb{R}^n$, then n is even and there is a linear change of coordinates carrying ω to the standard symplectic form.*

The above proposition is purely a linear algebra problem which we won't touch in this note. So, for us, Darboux's theorem will refer to the following.

Theorem 8 (Darboux)**.** *Let ω be a symplectic form on $\mathbb{R}^n$. Then, locally near the origin, there exists a smooth change of coordinates which carries ω to a constant 2-form.*

It will be useful to have an algebraic way of recognizing when a 2-form on $\mathbb{R}^n$ is constant. This is taken up in the next section.

2 The Euler vector field

The *Euler vector field* on $\mathbb{R}^n$ is the following linear vector field.

$$\mathcal{E} = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$$

The Euler vector field is invariant under linear changes of coordinates. To see this quickly, one may note that the flow generated by $\mathcal{E}$ is just scalar multiplication $(t, x) \mapsto e^t x$. Since this flow is invariant under linear changes of coordinates, so is $\mathcal{E}$.

The Euler vector field can be used to detect which smooth functions are homogeneous polynomials, or even when the coefficients of a tensor are homogeneous polynomials.

Proposition 9.

1. *A function f on $\mathbb{R}^n$ is a homogeneous polynomial of degree k if and only if $\mathcal{E}f = kf$.*
2. *A vector field X on $\mathbb{R}^n$ is linear if and only if $[\mathcal{E}, X] = 0$.*
3. *A k-form α on $\mathbb{R}^n$ is constant if and only if $\mathcal{L}_{\mathcal{E}}(\alpha) = k\alpha$, where $\mathcal{L}_X$ denotes Lie derivative with respect to a vector field X.*

We are only interested in the third part of the above proposition, so that is all we prove.

Proof. Recall that Lie differentiation of forms is an (ungraded) derivation for the wedge product which commutes with the exterior derivative.

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta) \qquad\qquad [\mathcal{L}_X, d] = 0 \tag{1}$$

The properties (1) can be obtained as formal consequences of *Cartan's Formula*, which expresses the Lie derivative in terms of exterior differentiation and contraction by X

$$\mathcal{L}_X = \iota_X d + d\iota_X$$

together with the fact that ι_X and d are graded derivations.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta \qquad \iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \iota_X\beta$$

Since $\mathcal{E}(x_i) = x_i$ for each coordinate function i (indeed, $\mathcal{E}$ leaves precisely the linear functionals invariant), the properties (1) imply that

$$\mathcal{L}_{\mathcal{E}}(dx^{i_1} \cdots dx^{i_k}) = kdx^{i_1} \cdots dx^{i_k}$$

and so, writing a generic k-form as ω as

$$\omega = \sum_{i_1 < \ldots < i_k} f_{i_1,\ldots,i_k} dx^{i_1} \cdots dx^{i_k},$$

we calculate that

$$\begin{aligned}
\mathcal{L}_{\mathcal{E}}(\omega) = k\omega &\iff \sum_{i_1 < \ldots < i_k} \mathcal{E}(f_{i_1,\ldots,i_k}) dx^{i_1} \cdots dx^{i_k} = 0 \\
&\iff \mathcal{E}(f_{i_1,\ldots,i_k}) = 0 \text{ for all } i_1 < \ldots < i_k \\
&\iff f_{i_1,\ldots,i_k} \text{ is a constant function for all } i_1 < \ldots < i_k.
\end{aligned}$$

$\square$

3 Euler-like vector fields

If X is a (smooth) vector field on $\mathbb{R}^n$ with a zero at the origin, we may write

$$X = L + \text{higher order terms}$$

where L is a linear vector field. In various applications, one would like to say that L tells us something about the behaviour of X near the origin. The best case scenario is that X is *linearizeable* in the sense that there is some smooth change of coordinates, fixing $x = 0$, which (locally) conjugates X to L.

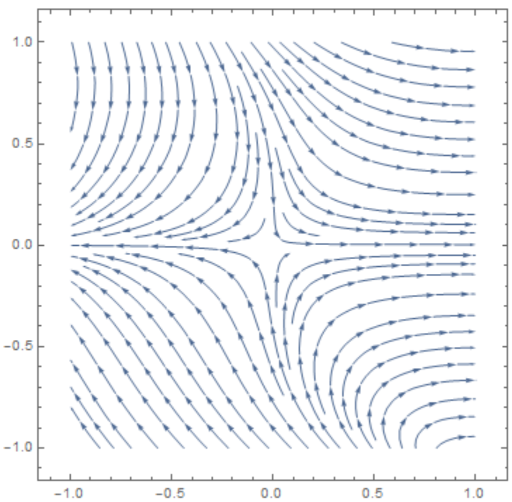

Figure 2: A nonlinear vector field

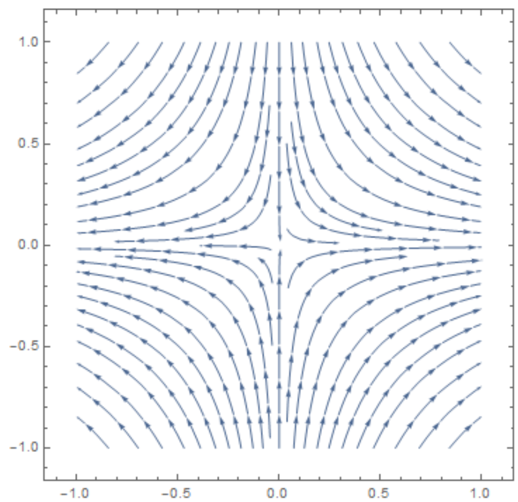

Figure 3: Its linear part

Obviously not every vector field is linearizeable. For example, the linear part of the 1-dimensional vector field $x^2\frac{d}{dx}$ is zero, and so is certainly not conjugate to the original vector field. So, at least one should stick to the case where the linear part L is nondegenerate. There is a quite general *Sternberg linearization theorem* which give as a sufficient condition for linearizability that the eigenvalues of the linear part are "non-resonant". In cases where Sternberg's theorem does not apply, the linearization question can be subtle. For example:

Example 10. $2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial y}$ is linearizeable and $2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial x}$ is not linearizeable, even though both of them have the same linear part $2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

One case to which Sternberg's theorem *does* apply is the case of an "Euler-like" vector field. Moreover, we can establish the linearizability of such vector fields without recourse to Sternberg's theorem.

Definition 11. A vector field X on $\mathbb{R}^n$ with a zero at the origin is called **Euler-like** if its linear part is the Euler vector field, i.e. if we can write $X = \mathcal{E} + \text{higher order terms}$.

Example 12. $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + xy\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y}$ is an Euler-like vector field on $\mathbb{R}^2$.

Theorem 13. *An Euler-like vector field X on $\mathbb{R}^n$ is linearizable, i.e. there exists a smooth, local change of coordinates fixing the origin and carying X to $\mathcal{E}$.*

Proof. Let X be a vector field on $\mathbb{R}^n$ with a zero at $x = 0$. Expand X as

$$X = L + Q + \text{remainder}$$

where L is a linear vector field and Q is a quadratic vector field. From Taylor's theorem, it follows that

$$X_t(x) = \begin{cases} \frac{1}{t}X(tx) & t \neq 0 \\ L(x) & t = 0 \end{cases} \qquad\qquad Y_t(X) = \begin{cases} \frac{1}{t^2}(X - L)(tx) & t \neq 0 \\ Q(x) & t = 0 \end{cases}$$

define smooth 1-parameter families of vector fields. As t changes from 1 to 0, the family X_t changes from X to L and the family Y_t changes from $X - L$ to Q.

To attack the linearization problem for X, an optimist might hope for a smooth 1-parameter family Ψ_t of diffeomorphisms, fixing the origin, with the property that $(\Psi_t)_*(X) = X_t$ for all $t \in [0, 1]$. In particular, Ψ_0 will give the linearization. Note that the rescaling diffeomorphisms

$$\kappa_t(x) = \tfrac{1}{t}x \qquad\qquad t > 0$$

satisfy $(\kappa_t)_*(X) = X_t$, essentially by definition, but sadly κ_0 does not make sense. We now assume X is Euler-like, so that $L = \mathcal{E}$ is the Euler vector field. The plan is to tweak the family κ_t a bit so that it makes sense at $t = 0$. Let Φ be the flow of X, and define

$$\Psi_t = \kappa_t \circ \Phi_{\log(t)} \qquad\qquad t > 0.$$

Note that $(\Psi_t)_*(X) = X_t$ still holds for $t \neq 0$ (since a vector field is invariant under is own flow). The rationale behind this choice is that, in the case where X is actually equal to $\mathcal{E}$, the flow is just $\Phi_t(x) = e^t x$ and so $\Psi_t(x) = \mathrm{id}$ for all $t > 0$, whence the family Ψ_t can trivially be continued to $t = 0$. We now show that Ψ_t can be continued when X is merely Euler-like. This we shall prove by checking that Ψ_t coincides with the flow of the time-dependent vector field Y_t, defined above. Indeed, loosely identifying the tangent spaces of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself so that $\mathcal{E}$ may be though of as the identity map $\mathbb{R}^n \to \mathbb{R}^n$, we calculate

$$\tfrac{d}{dt}\Psi_t(x) = \tfrac{d}{dt}\left(\frac{\Phi_{\log(t)}(x)}{t}\right) = \frac{X(\Phi^{\log(t)}(x))}{t^2} - \frac{\Phi^{\log(t)}(x)}{t^2} = \frac{X(t\Psi^t(x))}{t^2} - \frac{\mathcal{E}(\Psi_t(x))}{t} = Y_t(\Psi_t(x)).$$

Thus, we may continue the family Ψ_t to $t = 0$. Since $(\Psi_t)_*(X) = X_t$ holds for $t > 0$, it holds as well for $t = 0$ by continuity, and so $(\Psi_0)_*(X) = X_0 = \mathcal{E}$, and we have achieved the desired linearization of X. $\square$

4 Proof of Darboux's theorem

Finally, we come to the proof of Darboux's theorem, which is quite short given our preparations. Let us summarize the main points so far.

- For us, "Darboux's theorem" refers to the assertion "locally, every symplectic form on $\mathbb{R}^n$ can be changed into a constant 2-form after some smooth change of coordinates."
- A 2-form ω is constant if and only if $\mathcal{L}_{\mathcal{E}}(\omega) = 2\omega$ (Proposition 9, Part 3).
- Locally near the origin, every Euler-like vector can be changed into the actual Euler vector field after some smooth change of coordinates.

Thus, Darboux's theorem will be proved once we establish the following proposition.

Proposition 14. *If ω is a symplectic form on $\mathbb{R}^n$, then there exists a vector field X satisfying*

1. $\mathcal{L}_X(\omega) = 2\omega$.
2. *X is Euler-like.*

The point is that, after a change of coordinates, X becomes $\mathcal{E}$ and then (1) tells us that ω is constant in the new coordinate system. We conclude with the proof of this proposition.

Proof. Finding an X such that (1) alone is satisfied is quite straightforward (we can actually find X such that $\mathcal{L}_X(\omega)$ is any closed 2-form we desire), but in order to make (2) hold we should be a bit more particular in our choice. Let $\omega(0) = \sum_{i<j} c_{ij} dx^i dx^j$ be the constant part of ω so that

$$\omega = \sum_{i<j} c_{ij} dx^i dx^j + \text{closed 2-form vanishing at 0.}$$

Since we are working on $\mathbb{R}^n$, the Poincaré lemma ensures that ω has a primitive, i.e. a 1-form α such that $d\alpha = \omega$. We choose the obvious primitive $\sum_{i<j} c_{ij} x^i dx^j$ for the constant part. For the remainder term, we can choose a primitive which vanishes to one more order:

$$\alpha = \sum_{i<j} c_{ij} x^i dx^j + \text{1-form vanishing to 2nd order at 0.}$$

Since ω is nondegenerate, any 1-form arises as the contraction of ω by a unique vector field. We define X to be the unique vector field such that $\iota_X(\omega) = 2\alpha$. Then, from Cartan's formula and closedness of ω, we have $\mathcal{L}_X(\omega) = \iota_X d\omega + d\iota_X \omega = 0 + 2d\alpha = 2\omega$, as desired. A short calculation shows that $\iota_{\mathcal{E}}(\sum_{i<j} c_{ij} dx^i dx^j) = 2\sum_{i<j} c_{ij} x^i dx^j$ so, in the absence of error terms, X must be the Euler vector field. A bit of thought reveals that, if we put back the error terms, then X will be Euler-like:

$$X = \mathcal{E} + \text{vector field vanishing to 2nd order at 0.}$$

□