Introduction to C*-algebra homology theories

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September 5, 2018

Abstract

We examine the axioms for an abstract C*-algebra homology theory and draw out some of the standard consequences. Especially, we show that stable theories necessarily satisfy Bott periodicity using an infinite swindle argument due to Cuntz.

1 Introduction

The most prominent examples of homology theories for C*-algebras are K-theory and related theories. In this document, however, we will rarely be concerned with the inner workings of any particular theory—only their abstract homological properties. We begin by setting down axioms for a C*-algebra homology theory. Next, we show that, in any such theory, following the construction of the Barratt-Puppe sequence in homotopy theory, the higher homology groups and boundary maps can recovered from the 0th homology groups using iterated suspension and mapping cones. Next, we show that, in fact, any homotopy-invariant, half-exact functor from C*-algebras to abelian groups can be prolonged to a homology theory by the same methods. Lastly, we show that the addition of the stability axiom leads automatically to Bott periodicity using an argument originally due to Joachim Cuntz and appearing on pages 61-63 of [3]. Much of this exposition is based on Lecture 8 of [4], a set of course notes written by John Roe and available online through the AMS Open Math Notes project. See also [1], [3], [5] and [6].

2 Axioms

We begin with a rather spartan definition of a C*-algebra homology theory, requiring only homotopy-invariance and the existence of long exact sequences. Note that K-theory and its affiliates are furthermore stable theories, i.e. are invariant under tensoring with the compact operators. Stability has no analog among the Eilenberg-Steendrod axioms and will not be required for the time being, but is nonetheless a very important property of K-theory and leads to Bott periodicity, as explained in Section 8.

Definition 1. A *homology theory* for C*-algebras consists of

- a sequence of functors E_n, $n \leq 0$ from C*-algebras to abelian groups and
- for each short exact sequence of C*-algebras $0 \to I \to A \to B \to 0$, naturally induced boundary maps $\partial : E_{n-1}(B) \to E_n(I)$, $n \leq 0$

such that the following axioms are satisfied:

- *Homotopy invariance.* If φ and ψ are homotopic $*$-homomorphisms $A \to B$, then $E_n(\varphi) = E_n(\psi)$, $n \leq 0$.
- *Long exact sequences.* For each short exact sequence $0 \to I \to A \to B \to 0$, the functorially induced maps and boundary maps fit into a long exact sequence.

$$
\begin{tikzcd}
E_0(I) \arrow[r] & E_0(A) \arrow[r] & E_0(B) \\
E_{-1}(I) \arrow[r] & E_{-1}(A) \arrow[r] & E_{-1}(B) \arrow[ull, "\partial"] \\
E_{-2}(I) \arrow[r] & E_{-2}(A) \arrow[r] & E_{-2}(B) \arrow[ull, "\partial"] \\
 & \vdots \arrow[ul, "\partial"] &
\end{tikzcd}
$$

Above, when we say that the boundary maps are "natural", we mean that, given a commuting diagram

$$
\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I' \arrow[r] & A' \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}
$$

with exact rows, all of the resulting squares

$$
\begin{tikzcd}
E_{n-1}(B) \arrow[r, "\partial"] \arrow[d] & E_n(I) \arrow[d] \\
E_{n-1}(B') \arrow[r, "\partial"] & E_n(I')
\end{tikzcd}
$$

are commutative. Thus, a morphism of short exact sequences induces a morphism of long exact sequences.

3 Additivity

Next, we address the compatibility of homology theories with direct sums. Indeed, thanks to the long exact sequence axiom, the functors E_n of a homology theory are *half-exact*. That is, if $0 \to I \to A \to B \to 0$ is an exact sequence of C*-algebras, then $E_n(I) \to E_n(A) \to E_n(B)$ is exact in the middle. It is simple to see that, more generally, any half-exact functor is compatible with direct sums.

Lemma 2. *Let E be a half-exact functor from C*-algebras to abelian groups. Then,*

1. *For any C*-algebras A_1 and A_2, there is a unique identification of $E(A_1 \oplus A_2)$ with $E(A_1) \oplus E(A_2)$ under which the standard inclusions and projections of $A_1 \oplus A_2$ induce the standard inclusions and projections of $E(A_1) \oplus E(A_2)$.*
2. *Let $\varphi : A_1 \oplus A_2 \to B$ be a $*$-homomorphism and write it as $\varphi = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}$ where φ_i is the restriction of φ to A_i. Then, under the identification of $E(A_1 \oplus A_2)$ with $E(A_1) \oplus E(A_2)$, we have $E(\varphi) = \begin{bmatrix} E(\varphi_1) & E(\varphi_2) \end{bmatrix}$.*
3. *Relating to (2), if α and β are orthogonal $*$-homomorphisms $A \to B$ (so that $\alpha + \beta$ is also a $*$-homomorphism), then $E(\alpha + \beta) = E(\alpha) + E(\beta)$.*

Proof sketch.

1. Half-exactness gives $\mathrm{ran}(E(\iota_i)) = \ker(E(\mathrm{pr}_i))$ and functoriality gives $E(\mathrm{pr}_i)E(\iota_i) = \mathrm{id}_{E(A_i)}$.
2. By (1), we have $E(\iota_i) = \iota_i$, under the identification of $E(A_1 \oplus A_2)$ with $E(A_1) \oplus E(A_2)$, and so $E(\varphi_i) = E(\varphi \circ \iota_i) = E(\varphi) \circ \iota_i$.
3. Factor through the copy of $\alpha(A) \oplus \beta(A)$ in B.

□

Additivity can be leveraged to establish all sorts of expected properties of a homology theory. For example, we have the following proposition. We use the notation $SA = C_0(0,1) \otimes A$ for the *suspension* of a C*-algebra A.

Proposition 3. *Let A be a C*-algebra and let $r : SA \to SA$ be the isomorphism which reverses the interval, i.e. sends $f \in SA$ to $t \mapsto f(1-t)$. Then, for any homotopy-invariant, half-exact functor E from C*-algebras to abelian groups, $E(r) = -\mathrm{id}$.*

Proof. Define $\alpha, \beta : SA \to SA$ by

$$\alpha(f)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ 0 & \frac{1}{2} \le t \le 1 \end{cases} \qquad \beta(f)(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ f(2-2t) & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that α and β are orthogonal, so $\alpha + \beta$ is a $*$-homomorphism. One may check that α is homotopic to id, β is homotopic to r, and $\alpha + \beta$ is homotopic to 0. Thus, according to Lemma 2, Part 3, we have $\mathrm{id} + E(r) = E(\alpha) + E(\beta) = E(\alpha + \beta) = 0$, as desired. □

Later on in this exposition, we will be forced to make certain (arbitrary) choices of sign conventions. The above proposition has the consequence that, if we chose, we could hide these minus signs by reversing the orientation of an interval at some stage of our constructions. However, we choose not to resort to such trickery.

Let us also make a note of the following simple consequence of the long exact sequence axiom and the splitting lemma for abelian groups.

Proposition 4. *Let $0 \to I \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ be a short exact sequence of C*-algebras admitting a right split $\beta : B \to A$. Then, for any homology theory, there is a unique identification of $E_n(A)$ with $E_n(I) \oplus E_n(B)$, $n \leq 0$ under which ι, π and β induce the anticipated inclusions and projections.*

4 Recovering the higher groups using suspension

The *cone* $CA = C_0(0,1] \otimes A$ of a C*-algebra A is always contractible, so, in any homology theory, each of the boundary maps in the long exact sequence associated to the extension $0 \to SA \to CA \to A \to 0$ is an isomorphism.

$$\epsilon_A : E_{n-1}(A) \xrightarrow{\cong} E_n(SA)$$

Note that a $*$-homomorphism $\varphi : A \to B$ leads to a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SA & \longrightarrow & CA & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S\varphi} & & \downarrow{\scriptstyle C\varphi} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & SB & \longrightarrow & CB & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

and so, by naturality of boundary maps, a commuting square

$$\begin{array}{ccc} E_{n-1}(A) & \xrightarrow{\epsilon_A} & E_n(SA) \\ \downarrow{\scriptstyle E_{n-1}(\varphi)} & & \downarrow{\scriptstyle E_n(S\varphi)} \\ E_{n-1}(B) & \xrightarrow{\epsilon_B} & E_n(SB). \end{array}$$

Thus, the isomorphisms ϵ_A are natural in A. We record this as a proposition.

Proposition 5. *For any homology theory, the boundary maps of all exact sequences of the form $0 \to SA \to CA \to A \to 0$ give natural isomorphisms $\epsilon : E_{n-1} \to E_n \circ S$, $n \leq 0$.*

By the above, when dealing with a homology theory, one can rather harmlessly assume that $E_{-n} = E_0 \circ S^n$, $n \geq 0$.

5 Recovering the boundary maps using mapping cones

Let $0 \to I \to A \xrightarrow{\pi} B \to 0$ be a short exact sequence of C*-algebras. By definition, the *mapping cone* C_π of π is the C*-algebra

$$C_\pi = \{(f, a) \in CB \oplus A : f(1) = \pi(a)\},$$

where $CB = C_0(0, 1] \otimes B$ denotes the *cone* of B. In the commutative realm, where B should be thought of as a closed subset of A, the mapping cone corresponds to the space obtained by attaching the cone CB to A along B. A basic principle of homotopy theory is that coning off

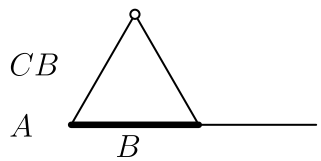


Figure 1: The mapping cone C_π.

a subset is as good as removing it, since the cone can be contracted. Thus, one should expect the mapping cone to be equivalent, for homotopy-theoretic purposes, to I which corresponds to the "complement" of B in A. Indeed, we have an obvious short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota_2} C_\pi \xrightarrow{\mathrm{pr}_1} CB \longrightarrow 0,$$

whence, CB being contractible, the inclusion of I as an ideal in C_π induces isomorphisms $E_n(I) \to E_n(C_\pi)$, $n \leq 0$ in any homology theory.

Another short exact sequence involving C_π is

$$0 \longrightarrow SB \xrightarrow{\iota_1} C_\pi \xrightarrow{\mathrm{pr}_2} A \longrightarrow 0.$$

Because $E_n(SB) \cong E_{n-1}(B)$ and $E_n(C_\pi) \cong E_n(I)$, the inclusion $\iota_1 : SB \to C_\pi$ leads to maps $E_{n-1}(B) \to E_n(I)$. It turns out that, up to a sign, these are the boundary maps of the exact sequence $0 \to I \to A \xrightarrow{\pi} B \to 0$.

Proposition 6. *Let $0 \to I \to A \xrightarrow{\pi} B \to 0$ be a short exact sequence of C*-algebras. Then, in any homology theory, the boundary maps $\partial : E_{n-1}(B) \to E_n(I)$, $n \leq 0$ are given by*

$$\partial = -E_n(\iota_2)^{-1} \circ E_n(\iota_1) \circ \epsilon_B,$$

where ι_1 and ι_2 are the factor inclusions of SB and I into C_π and $\epsilon_B : E_{n-1}(B) \xrightarrow{\cong} E_n(SB)$ is the boundary map of $0 \to SB \to CB \to B \to 0$.

Proof. The goal is to show that the square

$$\begin{tikzcd} E_{n-1}(B) \arrow[r, "\partial"] \arrow[d, "\epsilon_B"] & E_n(I) \arrow[d, "E_n(\iota_2)"] \\ E_n(SB) \arrow[r, "E_n(\iota_1)"] & E_n(C_\pi) \end{tikzcd}$$

commutes after a sign change. Let $\iota = \begin{bmatrix} \iota_1 & \iota_2 \end{bmatrix}$ be the inclusion of $SB \oplus I$ into C_π. Note $SB \oplus I$ sits as the kernel of the projection $p : C_\pi \mapsto B$ sending $(f, a) \mapsto f(1) = \pi(a)$. Consider the following three terms in the long exact sequence associated to $0 \to SB \oplus I \xrightarrow{\iota} C_\pi \xrightarrow{p} B \to 0$, writing d for the boundary map.

$$E_{n-1}(B) \xrightarrow{d} E_n(SB \oplus I) \xrightarrow{E_n(\iota)} E_n(C_\pi) \tag{1}$$

We claim that, after identifying $E_n(SB \oplus I)$ with $E_n(B) \oplus E_n(I)$, as in Lemma 2, the sequence (1) becomes

$$E_{n-1}(B) \xrightarrow{\begin{bmatrix} \epsilon_B \\ \partial \end{bmatrix}} E_n(SB) \oplus E_n(I) \xrightarrow{\begin{bmatrix} E_n(\iota_1) & E_n(\iota_2) \end{bmatrix}} E_n(C_\pi)$$

whence the equation $E_n(\iota) \circ d = 0$ reads $E_n(\iota_1) \circ \epsilon_B + E_n(\iota_1) \circ \partial = 0$, exactly the thing to be proved. Lemma 2, Part 2 immediately gives that $E_n(\iota) = \begin{bmatrix} E_n(\iota_1) & E_n(\iota_2) \end{bmatrix}$ under the identification. To see that the boundary map d in (1) has the desired form, consider the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & SB \arrow[r] & CB \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & SB \oplus I \arrow[r, "\iota"] \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & C_\pi \arrow[r, "p"] \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & B \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & A \arrow[r, "\pi"] & B \arrow[r] & 0 \end{tikzcd}$$

Naturality of boundary maps tells us that

$$\begin{tikzcd} E_{n-1}(B) \arrow[r, "\epsilon_B"] & E_n(SB) \\ E_{n-1}(B) \arrow[r, "d"] \arrow[u, equal] \arrow[d, equal] & E_n(SB \oplus I) \arrow[u, "E_n(\mathrm{pr}_1)"'] \arrow[d, "E_n(\mathrm{pr}_2)"] \\ E_{n-1}(B) \arrow[r, "\partial"] & E_n(I) \end{tikzcd}$$

is commutative which, after the identification $E_n(SB \oplus I) = E_n(SB) \oplus E_n(I)$, says exactly that the components of d are ϵ_B and ∂. $\square$

6 Equivalence of mapping cone and ideal, revisited

Let $0 \to I \to A \xrightarrow{\pi} B \to 0$ be a short exact sequence of C*-algebras. In the preceding section, we noted that the inclusion $\iota_2 : I \to C_\pi$ induces isomorphisms on any homology theory thanks to the short exact sequence $0 \to I \to C_\pi \to CB \to 0$ and the long exact sequence axiom. We need to slightly generalize this observation so that it applies to any homotopy-invariant, half-exact functor.

Lemma 7. *Let $0 \to I \to A \xrightarrow{\pi} B \to 0$ be a short exact sequence of C*-algebras. Then, for any homotopy-invariant, half-exact functor E from C*-algebras to abelian groups, the inclusion $\iota_2 : I \to C_\pi$ induces an isomorphism $E(I) \to E(C_\pi)$.*

Proof. Applying half-exactness of E to the short exact sequence $0 \to I \to C_\pi \to CB \to 0$, we get that $E(\iota_2)$ is surjective. For injectivity, it is useful to introduce the C*-algebra $Q = \{f \in C[0,1] \otimes A : f(0) \in I\}$, of which I is a deformation retract. There are maps

$$\alpha : I \to Q \qquad\qquad \beta : Q \to I \qquad\qquad \gamma : Q \to C_\pi$$

given as follows: α sends an element of I to the corresponding constant function, β is $f \mapsto f(0)$ and γ is $f \mapsto (\pi \circ f, f(1))$. Obviously, $\beta \circ \alpha = \mathrm{id}_I$. Furthermore, it is easy to check that $\alpha \circ \beta$ is homotopic to id_Q, so α and β induce inverse isomorphisms between $E(I)$ and $E(Q)$. The kernel of γ is a copy of CI so, applying half-exactness to the short exact sequence $0 \to CI \to Q \xrightarrow{\gamma} C_\pi \to 0$, we see that $E(\gamma)$ is injective. Finally, the diagram

$$\begin{tikzcd} I \arrow[r, "\iota_2"] \arrow[d, "\alpha"] & C_\pi \\ Q \arrow[ur, "\gamma"] & \end{tikzcd}$$

is commutative so, from $E(\iota_2) = E(\gamma) \circ E(\alpha)$, we get surjectivity of $E(\iota_2)$. $\square$

7 The Barratt-Puppe sequence

Our next task is to take any half-exact, homotopy-invariant functor E from C*-algebras to abelian groups and construct a containing homology theory with $E_0 = E$. According to Proposition 5, if this can be done at all, it can be done while setting $E_{-n} = E \circ S^n$. Since the functor S preserves exact sequences and homotopy of $*$-homomorphisms, the functors E_n so defined are also half-exact and homotopy-invariant.

Coming to the boundary maps, there is a bit of choice in how to proceed. By Proposition 5, the boundary maps of exact sequences $0 \to SA \to CA \to A \to 0$ are going to be natural automorphisms $\epsilon_A : E_{n-1}(A) = E_n(SA) \to E_n(SA)$. It is reasonable to impose either $\epsilon_A = \mathrm{id}$ or $\epsilon_A = -\mathrm{id}$. In the first case, according to Proposition 6, the boundary maps

$$\partial : E_{n-1}(B) = E_n(SB) \to E_n(I)$$

of all other exact sequences $0 \to I \to A \xrightarrow{\pi} B \to 0$ must be defined by

$$\partial = -E_n(\iota_2)^{-1} \circ E_n(\iota_1),$$

where ι_1 and ι_2 are the factor inclusions of SB and I into C_π. Alternatively, we could impose $\epsilon_A = -\mathrm{id}$ for all A, leading to a sign-free version of the above formula for the boundary maps. We shall opt for the first option and accept the unpleasant minus sign. By Proposition 3, we could also hide this minus sign by reversing the orientation of an interval somewhere.

Theorem 8. *Let E be a homotopy-invariant, half-exact functor from C*-algebras to abelian groups. Define a sequence of functors E_{-n}, $n \geq 0$ by iterated suspension: $E_{-n} = E \circ S^n$. For each short exact sequence of C*-algebras $0 \to I \to A \xrightarrow{\pi} B \to 0$, define boundary maps $\partial : E_{n-1}(B) = E_n(SB) \to E_n(I)$ by $\partial = -E_n(\iota_2)^{-1} \circ E_n(\iota_1)$, where ι_1 and ι_2 are the factor inclusions of SB and I into the mapping cone C_π. Then, these functors and boundary maps define a C*-algebra homology theory.*

Proof. Since suspension preserves exact sequences and homotopy of $*$-homomorphisms, the functors E_n are homotopy-invariant and half-exact. Naturality of the boundary maps follows from a naturality in the mapping cone construction: given a commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & A \arrow[r, "\pi"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I' \arrow[r] & A' \arrow[r, "\pi'"] & B' \arrow[r] & 0,
\end{tikzcd}$$

we obtain a corresponding commuting diagram

$$\begin{tikzcd}
SB \arrow[r, "\iota_1"] \arrow[d] & C_\pi \arrow[d] & I \arrow[l, "\iota_2"'] \arrow[d] \\
SB' \arrow[r, "\iota_1'"] & C_{\pi'} & I' \arrow[l, "\iota_2'"']
\end{tikzcd}$$

from which the naturality of the boundary maps can be deduced. So, the main thing is to check that, given any short exact sequence of C*-algebras $0 \to I \to A \xrightarrow{\pi} B \to 0$, the associated long exact sequence really is exact. That is, for any $n \leq 0$,

$$E_n(SA) \longrightarrow E_n(SB) \xrightarrow{\partial} E_n(I) \longrightarrow E_n(A) \longrightarrow E_n(B) \qquad (2)$$

is exact. Since the functors E_n are half-exact, (2) is exact at $E_n(A)$. By definition of the boundary map,

$$\begin{tikzcd}
E_n(SB) \arrow[r, "\partial"] \arrow[d, "-\mathrm{id}"] & E_n(I) \arrow[r] \arrow[d, "E_n(\iota_2)"] & E_n(A) \arrow[d, equal] \\
E_n(SB) \arrow[r, "E_n(\iota_1)"] & E_n(C_\pi) \arrow[r, "E_n(\mathrm{pr}_2)"] & E_n(A)
\end{tikzcd}$$

is a commutative diagram and so, the vertical maps being isomorphisms and the lower row being exact because $0 \to SB \overset{\iota_1}{\to} C_\pi \overset{\mathrm{pr}_2}{\to} A \to 0$ is an exact sequence, we get that (2) is exact at $E_n(I)$. Finally, to get exactness at $E_n(SB)$, define a C*-algebra

$$D = \{(f, g) \in CB \oplus CA : f(1) = \pi(g(1))\},$$

The algebra D is a realization of the mapping cone of $0 \to SB \to C_\pi \to A \to 0$. Accordingly,

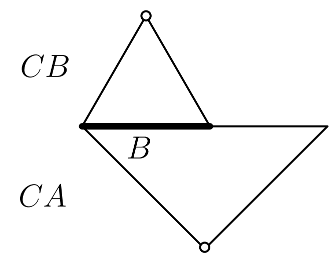


Figure 2: The double cone D.

D fits into an exact sequence

$$0 \longrightarrow SA \xrightarrow{\iota_2} D \xrightarrow{q} C_\pi \longrightarrow 0$$

where q sends $(f, g) \mapsto (f, f(1)) = (f, \pi(g(1)))$ and furthermore, by Lemma 7, the inclusion $\iota_1 : SB \to C_\pi$ induces an isomorphism on E_n. We argue that the diagram

$$\begin{tikzcd}
E_n(SA) \arrow[r, "E_n(S\pi)"] \arrow[d, equal] & E_n(SB) \arrow[r, "\partial"] \arrow[d, "-E_n(\iota_1)"] & E_n(I) \arrow[d, "E_n(\iota_2)"] \\
E_n(SA) \arrow[r, "E_n(\iota_2)"] & E_n(D) \arrow[r, "E_n(q)"] & E_n(C_\pi)
\end{tikzcd}$$

is commutative, which will give the exactness of (2) at $E_n(SB)$. Firstly, $q \circ \iota_1 = \iota_1 : SB \to C_\pi$, so the square on the right commutes by definition of ∂. Coming to the square on the left, according to Proposition 3, the interval reversal map $r : SA \to SA$ induces $-\mathrm{id}$ on E_n, so we will be done if $\iota_2 \circ r : SA \to D$ and $\iota_1 \circ S\pi : SA \to D$ are homotopic $*$-homomorphisms. This is the case, the rough idea being to slide the map from one cone of D to the other. In terms of formulas, define $\varphi_s : SA \to D$, $s \in [0, 1]$ by $\varphi_s(f) = (g_s, h_s)$, where

$$g_s(t) = \begin{cases} 0 & 0 \le t \le s \\ \pi(f(t - s)) & s \le t \le 1 \end{cases} \qquad h_s(t) = \begin{cases} 0 & 0 \le t \le 1 - s \\ f(2 - s - t) & 1 - s \le t \le 1 \end{cases}$$

One may verify that $\varphi_0 = \iota_1 \circ S\pi$ and $\varphi_1 = \iota_2 \circ r$. $\square$

8 Bott Periodicity

In this section, we that prove every stable homology theory satisfies Bott periodicity following the proof given by Cuntz in [3]. Fundamentally, Cuntz's proof is a kind of infinite swindle. In its course, we shall need to show that a pair of $*$-homomorphisms induce the same map at the homological level. To achieve this, we pass to a pair of auxiliary $*$-homomorphisms by adding on an extra (orthogonal) piece to the original ones then show that these enlarged $*$-homomorphisms are actually homotopic. The remainder of the proof is then given over to carefully "cancelling off" the added piece (at the homological level) in order that the original $*$-homomorphisms may be seen to also induce the same map. This last stage of the proof is somewhat delicate, but elementary; it makes use of a purpose-built split extension of the Toeplitz algebra, constructed as the pullback of another extension. We write $\mathbb{K}$ for the C*-algebra of compact operators on $\ell^2(\mathbb{N})$ and $E_{i,j}$, $i, j \geq 0$ for the rank-1 operator therein which sends the jth standard basis vector to the ith. The same notation $E_{i,j}$ will be used for the matrix units in $M_n(\mathbb{C})$, $0 \leq i, j \leq n-1$.

Definition 9. A functor E from C*-algebras to abelian groups is called *stable* if, for every C*-algebra A, the corner inclusion $E_{0,0} \otimes \mathrm{id}_A : A \to \mathbb{K} \otimes A$ induces an isomorphism $E(A) \overset{\cong}{\to} E(\mathbb{K} \otimes A)$.

Because, in any homology theory, the functors E_{-n} are naturally isomorphic to $E_0 \circ S^n$ and stabilization commutes with suspension, if the initial functor E_0 of a homology theory is stable, so are the higher functors E_{-n}.

Cuntz's proof of Bott periodicity makes use of the *Toeplitz algebra* $\mathfrak{T}$, which we take to be the C*-algebra on $\ell^2(\mathbb{N})$ generated by the unilateral shift S which sends each standard basis vector to its successor: $\xi_i \mapsto \xi_{i+1}$, $i \geq 0$. The Toeplitz algebra is universal with respect to isometries: given an isometry W in a C*-algebra A, there is a unique $*$-homomorphism $\mathfrak{T} \to A$ sending $S \mapsto W$. Denote by $\sigma : \mathfrak{T} \to C(S^1)$ the *symbol map*, which sends $S \mapsto z$, the coordinate function of the unit circle $S^1 \subseteq \mathbb{C}$. The kernel of σ is the compact operators $\mathbb{K}$, giving the very important *Toeplitz extension.*

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

Denote by $\sigma_1 : \mathfrak{T} \to \mathbb{C}$ the composition of σ with $\mathrm{eval}_1 : C(S^1) \to \mathbb{C}$ i.e. the map sending $S \mapsto 1$. The Toeplitz algebra is unital, so we have also the scalar map $\iota : \mathbb{C} \to \mathfrak{T}$ determined by $1 \mapsto 1$. Our main goal is to provide a homotopical proof of the following statement, of which Bott periodicity is a straightforward consequence.

Theorem 10. *Let E_{-n}, $n \geq 0$ be a stable homology C*-algebra homology theory. Then, the maps $\sigma_1 : \mathfrak{T} \to \mathbb{C}$ and $\iota : \mathbb{C} \to \mathfrak{T}$ induce inverse isomorphisms between $E_{-n}(\mathfrak{T})$ and $E_{-n}(\mathbb{C})$. More generally, tensoring with any C*-algebra A, we get isomorphisms between $E_{-n}(\mathfrak{T} \otimes A)$ and $E_{-n}(A)$.*

The proof of the more general statement, involving an arbitrary algebra A, is not different from the proof in the special case $A = \mathbb{C}$. One simply needs to tensor algebras by A and maps by id_A in the appropriate places. Accordingly, we shall only bother to prove the first statement. In one direction, we have $\sigma_1 \circ \iota = \mathrm{id}_{\mathbb{C}}$, so one just needs to show that $\psi_1 = \iota \circ \sigma_1$, the homomorphism $\mathfrak{T} \to \mathfrak{T}$ sending $S \mapsto 1$, induces the identity on $E(\mathfrak{T})$. The simplest explanation for this would be that ψ_1 is homotopic to the identity map $\psi_0 = \mathrm{id}_{\mathfrak{T}}$, but this cannot be so. Indeed, a homotopy between ψ_0 and ψ_1 amounts to a path of isometries joining the two images $\psi_0(S) = S$ and $\psi_1(S) = 1$ of the generator of the Toeplitz algebra.[1] Because S has index -1, such a path does not exist.

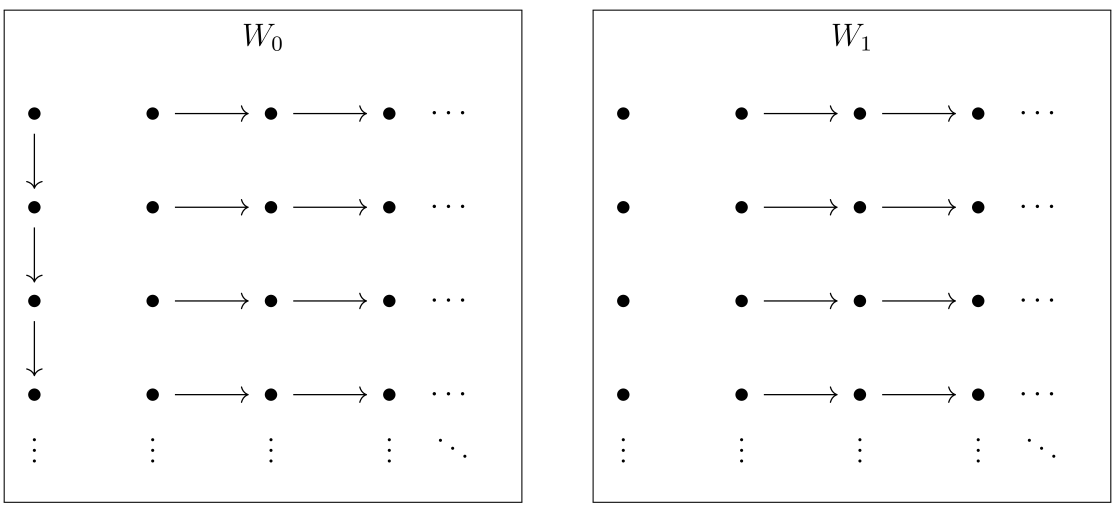
To get around this difficulty we work in the larger C*-algebra $A = \mathbb{K} \otimes \mathfrak{T} + \mathfrak{T} \otimes 1$, which acts on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. The algebra A fits into an extension

$$0 \longrightarrow \mathbb{K} \otimes \mathfrak{T} \longrightarrow A \xrightarrow{\pi} C(S^1) \longrightarrow 0 \tag{3}$$

where first map is inclusion of the ideal and π is determined by its restriction to $\mathfrak{T} \otimes 1$ where it acts as the obvious symbol map $\mathfrak{T} \otimes 1 \to C(S^1)$ sending $S \otimes 1 \mapsto z$. Define isometries $W_0, W_1 \in A$ by

$$W_0 = E_{0,0} \otimes S + S(1 - E_{0,0}) \otimes 1 \qquad\qquad W_1 = E_{0,0} \otimes 1 + S(1 - E_{0,0}) \otimes 1$$

whose action on the standard basis of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ is pictured below.



Cuntz's proof is based on the fact that these isometries are homotopic in A. Considering their behaviour on the left column, one may think of this as a sort of generalized homotopy between S and 1 in $\mathfrak{T}$. Thus, even though no honest homotopy can exist, by making some extra room and working with isometries whose defect projections are infinite-dimensional, they can be connected.

[1] Indeed, a legitimate way to define a homotopy between two $*$-homomorphisms $\psi_0, \psi_1 : A \to B$ is as a $*$-homomorphism $\Psi : A \to C([0,1]) \otimes B$ satisfying $\psi_i = \mathrm{eval}_i \circ \Psi$ for $i = 0, 1$. When $A = \mathfrak{T}$, the universal property of $\mathfrak{T}$ says that Ψ amounts to a choice of isometry in $C([0,1]) \otimes B$, which is the same thing as a norm-continuous path of isometries in B (assuming unital $*$-homomorphisms, for simplicity).

Lemma 11. *There is a path of isometries W_t, $t \in [0,1]$ in A from W_0 to W_1 which, moreover, satisfies $\pi(W_t) = z$ for all $t \in [0,1]$.*

The proof of this lemma will rest on a simple fact concerning permutations of a countably infinite set: namely that the 2-sided shift can be factored as the product of two involutions. Indeed, on $\mathbb{Z}$, composing $x \mapsto -x$ and $x \mapsto 1-x$ yields the shift $x \mapsto x+1$. Geometrically speaking, this is the infinite dihedral group version of the fact from plane geometry that the composition of two reflections is a rotation through twice the angle between the reflecting lines. Indeed, $x \mapsto -x$ and $x \mapsto 1-x$ are reflection through $x = 0$ and $x = \frac{1}{2}$, respectively, and $x \mapsto x+1$ is a sort of infinite rotation.

The factorization just discussed can be expatriated to $M_2(\mathbb{C}) \otimes \mathfrak{T}$, thought of as a C*-algebra on $\ell^2(\{0,1\}) \otimes \ell^2(\mathbb{N})$, where it assumes the form of a factorization of the unitary

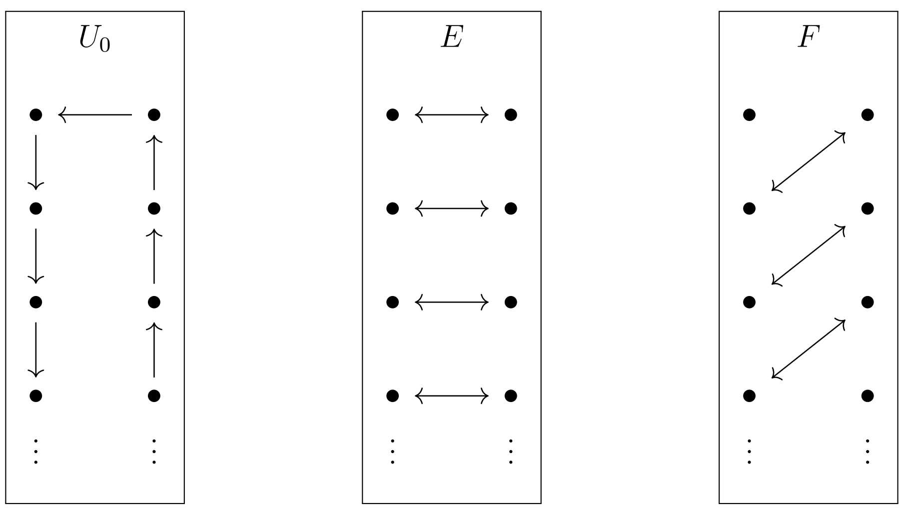
$$U_0 = E_{0,0} \otimes S + E_{0,1} \otimes E_{0,0} + E_{1,1} \otimes S^*$$

(a manifestation of the bilateral shift) as the product $U = FE$ of the two symmetries

$$E = E_{0,1} \otimes 1 + E_{1,0} \otimes 1 \qquad F = E_{0,0} \otimes E_{0,0} + E_{1,0} \otimes S^* + E_{0,1} \otimes S,$$

as may be read from Figure 3 below.

Figure 3: Factorization of the bilateral shift as a product of two symmetries.



Recall that a *symmetry* F in a unital C*-algebra A is an idempotent unitary: $F^2 = 1$, $F^* = F$. There is a bijective correspondence between projections and symmetries; every symmetry F takes the form $F = -P + P^\perp$ where P is a projection and $P^\perp = 1 - P$ is the orthocomplementary projection. This presentation makes it clear that every symmetry is connected to the identity by the path of unitaries $t \mapsto e^{\pi i t} P + P^\perp$. Thus, the factorization $U_0 = FE$ above shows that the bilateral shift U_0 is homotopic to 1 inside the unitary group of $M_2(\mathbb{C}) \otimes \mathfrak{T}$.

Proof of Lemma 11. In the above discussion, we noted there is a path U_t, $t \in [0,1]$ of unitaries in $M_2(\mathbb{C}) \otimes \mathfrak{T}$ from the bilateral shift $U_0 = E_{0,0} \otimes S + E_{0,1} \otimes E_{0,0} + E_{1,1} \otimes S^*$ to the identity $U_1 = E_{0,0} + E_{1,1}$. Let us now work in $A = \mathbb{K} \otimes \mathfrak{T} + \mathfrak{T} \otimes 1$ and add $(1 - E_{0,0} - E_{1,1}) \otimes 1$ to this homotopy, to get a path of unitaries V_t from $V_0 = U_1 + (1 - E_{0,0} - E_{1,1}) \otimes 1$ to $V_1 = 1$. Because the homotopy really takes place in $M_2(\mathbb{C}) \otimes \mathfrak{T} \subseteq \mathbb{K} \otimes \mathfrak{T}$, it is clear that $\pi(V_t) = 1$ for all $t \in [0,1]$. Now, setting $W_t = V_t W_1$ gives the desired path of isometries from W_0 to W_1 which furthermore satisfies $\pi(W_t) = \pi(V_t)\pi(W_1) = z$ for all t, as needed. □

It remains to see how Lemma 11 applies to prove Theorem 10. We package the remainder of the argument as the following technical lemma.

Lemma 12. *Suppose $0 \to I \to A \xrightarrow{\pi} B \to 0$ is a short exact sequence, $\varphi_i : C \to I$, $i = 0, 1$ and $\sigma : C \to B$ are $*$-homomorphisms, and σ has a lift $\overline{\sigma} : C \to A$.*

$$\begin{tikzcd}
 & & & C \arrow[lll, bend right, "\varphi_i"'] \arrow[dl, "\overline{\sigma}"'] \arrow[d, "\sigma"] & \\
0 \arrow[r] & I \arrow[r] & A \arrow[r, "\pi"] & B \arrow[r] & 0
\end{tikzcd}$$

If $\overline{\sigma}$ is orthogonal to φ_0 and φ_1, so that $\theta_i = \varphi_i + \overline{\sigma}$, $i = 0, 1$ are $*$-homomorphisms and there is a homotopy $(\theta_t)_{t\in[0,1]}$ joining θ_0 to θ_1 such that $\pi \circ \theta_t = \sigma$ for all $t \in [0,1]$, then φ_0 and φ_1 induce the same maps $E_n(C) \to E_n(I)$ in any C-algebra homology theory.*

Proof. The hypotheses are set up so that we can make use of the pullback C*-algebra $\overline{A} = \{(a, c) \in A \oplus C : \pi(a) = \sigma(c)\}$ which fits into a pulled back short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota_1"] \arrow[d, equal] & \overline{A} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r, "\pi"] & B \arrow[r] & 0.
\end{tikzcd}$$

The lift $\overline{\sigma}$ of σ yields a right split of the top sequence $\beta : C \to \overline{A}$ sending $c \mapsto (\overline{\sigma}(c), c)$. Thus, the induced maps $E_n(\iota_1) : E_n(I) \to E_n(\overline{A})$ are injective. In the same way, the maps θ_t, $t \in [0,1]$ (which are also lifts of σ) yield right splits $\chi_t : C \to \overline{A}$ sending $c \mapsto (\theta_t(c), c)$. By homotopy invariance, we have $E_n(\chi_0) = E_n(\chi_1)$. Writing $\chi_i = \beta_i + \varphi_i \circ \iota_1$ and using Proposition 2, we get $E_n(\beta) + E_n(\varphi_0) \circ E_n(\iota_1) = E_n(\beta) + E_n(\varphi_1) \circ E_n(\iota_1)$, whence, canceling $E_n(\beta)$ and using injectivity of $E_n(\iota_1)$, we get the result. □

Finally, we assemble Lemma 11 and Lemma 12 to prove Theorem 10.

Proof of Theorem 10. Since $\sigma_1 \circ \iota = \mathrm{id}_{\mathbb{C}}$, we only need to show that $\iota \circ \sigma_1$ induces the identity on $E_n(\mathfrak{T})$. Consider $\mathbb{K} \otimes \mathfrak{T}$ as a C*-algebra of operators on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Let $\varphi_0 : \mathfrak{T} \mapsto \mathbb{K} \otimes \mathfrak{T}$ denote the corner inclusion $E_{0,0} \otimes \mathrm{id}_{\mathfrak{T}}$ which sends the unilateral shift S to the partial isometry $E_{0,0} \otimes S$. Let $\varphi_1 = \varphi_1 \circ \iota \circ \sigma_1$, which sends S to the projection $E_{0,0} \otimes 1$. By the assumption that E_n is stable, we will be done if we show that $E_n(\varphi_0) = E_n(\varphi_1)$. We apply Lemma 12, taking $I = \mathbb{K}\otimes\mathfrak{T}$, $A = \mathbb{K}\otimes\mathfrak{T}+\mathfrak{T}\otimes 1$, $B = C(S^1)$, $0 \to I \to A \xrightarrow{\pi} B \to 0$ the exact sequence (3) from Lemma 11, $C = \mathfrak{T}$, σ the symbol map $\mathfrak{T} \to C(S^1)$ and the split $\overline{\sigma} : \mathfrak{T} \to B$ to be given by $S \mapsto S(1 - E_{0,0}) \otimes 1$.

$$
\begin{tikzcd}
 & & & \mathfrak{T} \arrow[dll, bend right, "\varphi_i"'] \arrow[dl, "\overline{\sigma}"'] \arrow[d, "\sigma"] & \\
0 \arrow[r] & \mathbb{K}\otimes\mathfrak{T} \arrow[r] & \mathbb{K}\otimes\mathfrak{T}+\mathfrak{T}\otimes 1 \arrow[r, "\pi"] & C(S^1) \arrow[r] & 0
\end{tikzcd}
$$

It is simple to check that the hypotheses of Lemma 12 are satisfied, and so φ_0 and φ_1 induce the same maps $E(\mathfrak{T}) \to E(\mathbb{K}\otimes\mathfrak{T})$, as was to be proved. $\square$

Define $\mathfrak{T}_0$ to be the kernel of the symbol map $\sigma_1 : \mathfrak{T} \to \mathbb{C}$, i.e. the C*-algebra of Toeplitz operators whose symbol vanishes at $1 \in S^1$. Indeed, $\mathfrak{T}$ is the unitization of $\mathfrak{T}_0$, the splitting being the map $\iota : \mathbb{C} \to \mathfrak{T}$.

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{T}_0 \arrow[r] & \mathfrak{T} \arrow[r, "\sigma_1"] & \mathbb{C} \arrow[l, bend right, "\iota"'] \arrow[r] & 0
\end{tikzcd}
$$

Thus, Theorem 10, together with Proposition 4, gives

Corollary 13. *For any stable homology theory, one has* $E_n(\mathfrak{T}_0) = 0$*,* $n \leq 0$ *and, more generally,* $E_n(\mathfrak{T}_0 \otimes A) = 0$*,* $n \leq 0$ *for any C*-algebra* A*.*

The above corollary quickly leads to Bott periodicity through consideration of the nonunital form of the Toeplitz extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{T}_0 \xrightarrow{\sigma_0} C_0(0,1) \longrightarrow 0$$

The map σ_0 is the usual symbol map, except that $(0,1)$ is identified with $S^1\backslash\{1\}$ via $t \mapsto e^{2\pi i t}$. More generally, because $C_0(0,1)$ is nuclear, this sequence remains exact after tensoring with an arbitrary C*-algebra A (see Corollary 3.7.4 in [2]).

$$0 \longrightarrow \mathbb{K}\otimes A \longrightarrow \mathfrak{T}_0 \otimes A \xrightarrow{\sigma_0\otimes\mathrm{id}} SA \longrightarrow 0$$

Corollary 13 shows that the middle terms of the long exact sequences associated to the above short exact sequence all vanish, for any stable homology theory, and so we get

Theorem 14 (Bott periodicity)**.** *For any stable homology theory, for any C*-algebra* A*, the boundary maps associated to the extension* $0 \to \mathbb{K}\otimes A \to \mathfrak{T}_0 \otimes A \xrightarrow{\sigma_0\otimes\mathrm{id}} SA \to 0$ *are natural isomorphisms* $E_{n-2}(A) \to E_n(A)$*, having made use of the natural identifications* $E_{n-1}(SA) = E_{n-2}(A)$ *and* $E_n(\mathbb{K}\otimes A) = E_n(A)$*.*

9 Relation to the usual Bott map in K-theory

In this last section we restrict attention to the usual K-theory functor and relate the isomorphism $K_0(S^2A) \to K_0(A)$ of Theorem 14 coming from the boundary map of the extension

$$0 \longrightarrow \mathbb{K} \otimes A \longrightarrow \mathfrak{T}_0 \otimes A \xrightarrow{\sigma_0 \otimes \mathrm{id}} SA \longrightarrow 0$$

to the Bott map $K_0(A) \to K_0(S^2A)$ coming from the tautological line bundle on $\mathbb{CP}^1$. It makes little difference if we restrict attention to the case $A = \mathbb{C}$, and we shall content ourselves with an analysis of this case. Since $K_0(S^2\mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$, only two isomorphisms are possible. Thus, the entirety of our reward for the expenditure of effort here will be the resolution of a sign ambiguity.

First, we should elect a generator for the K_0 group of $S^2\mathbb{C} = C_0(0,1)^2$. We use $[H] - [1]$, where H is, in effect, the tautological projection[2] in $M_2(C(\mathbb{CP}^1))$. We shall prefer to think of $\mathbb{CP}^1$ as the one-point compactification $\mathbb{D}^\circ \cup \{\infty\}$ of the open unit disk $\mathbb{D}^\circ = \{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, as the closed unit disk $\mathbb{D}$ with boundary circle S^1 collapsed to a point. In this picture, thinking of $C(\mathbb{CP}^1)$ as $\{f \in C(\mathbb{D}) : f \text{ is constant on } S^1\}$, the tautological bundle is given on $z \in \mathbb{D}$ by projection onto the unit vector $(z, \sqrt{1 - |z|^2})$.

$$H(z) = \begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \overline{z}\sqrt{1-|z|^2} & 1 - |z|^2 \end{bmatrix}$$

To realize the generator $[H] - [1]$ as an element of $K_0(C_0(0,1)^2)$, we need to choose an orientation-preserving homeomorphism $(0,1)^2 \to \mathbb{D}^\circ$. Any two such homeomorphisms are isotopic, so the precise choice is not so important. Nonetheless, it is convenient for present purposes to use $(s,t) \mapsto (1-s) \cdot 1 + s \cdot e^{2\pi i t}$.

Proposition 15. *The isomorphism $K_0(C_0(0,1)^2) \to K_0(\mathbb{C})$ of Theorem 14 sends the generator $[H] - [1]$ of $K_0(C_0(0,1)^2)$ to $-[1] \in K_0(\mathbb{C})$.*

Proof. If we identify $K_0(\mathbb{C})$ with $K_0(\mathbb{K})$ using $[1] \mapsto [E_{0,0}]$, then the isomorphism of Theorem 14 is just the boundary map $\partial : K_0(C_0(0,1)^2) \to K_0(\mathbb{K})$ of the exact sequence $0 \to \mathbb{K} \to \mathfrak{T}_0 \overset{\sigma_0}{\to} C_0(0,1) \to 0$. By definition, $\partial = -\iota_{1*} \circ (\iota_{2*})^{-1}$ where ι_1 and ι_2 are the inclusions of $C_0(0,1)^2$ and $\mathbb{K}$ into the mapping cone C_{σ_0}. So, we just need to show that $\iota_{1*}([H] - [1]) = \iota_{2*}([E_{0,0}])$ or, equivalently,

$$\widetilde{\iota_{1*}}([H]) - \iota_{2*}([E_{0,0}]) = [1] \in K_0(\widetilde{C_{\sigma_0}}).$$

The unitization is necessary because H has coefficients in $C(\mathbb{CP}^1) = \widetilde{C_0(0,1)^2}$.

[2] The elements of $\mathbb{CP}^1$ are complex lines in $\mathbb{C}^2$, which may equally well be thought of as rank-1 projections in $M_2(\mathbb{C})$.

Let us take a moment to clarify what $\widetilde{C_{\sigma_0}}$ looks like, given all the implicit identifications we are making. By definition, C_{σ_0} sits in the pullback diagram

$$\begin{tikzcd} C_{\sigma_0} \arrow[r] \arrow[d] & \mathfrak{T}_0 \arrow[d, "\sigma_0"] \\ C_0((0,1] \times (0,1)) \arrow[r, "f \mapsto f(1,\cdot)"] & C_0(0,1). \end{tikzcd}$$

If we identify $(0,1] \times (0,1)$ with $\mathbb{D} \setminus \{1\}$ using $(s,t) \mapsto (1-s) \cdot 1 + s \cdot e^{2\pi i t}$ and $(0,1)$ with $S^1 \setminus \{1\}$ using $t \mapsto e^{2\pi i t}$, then this diagram becomes

$$\begin{tikzcd} C_{\sigma_0} \arrow[r] \arrow[d] & \mathfrak{T}_0 \arrow[d, "\sigma"] \\ C_0(\mathbb{D} \setminus \{1\}) \arrow[r, "\mathrm{restr}"] & C_0(S^1 \setminus \{1\}). \end{tikzcd}$$

where restr is the restriction map. Since the unitization of a pullback is the pullback of the unitized diagram, we get the that the unitization $\widetilde{C_{\sigma_0}}$ sits in the following pullback diagram

$$\begin{tikzcd} \widetilde{C_{\sigma_0}} \arrow[r] \arrow[d] & \mathfrak{T} \arrow[d, "\sigma"] \\ C(\mathbb{D}) \arrow[r, "\mathrm{restr}"] & C(S^1) \end{tikzcd} .$$

In this picture, the unitization of $\iota_1 : C_0(0,1)^2 \to C_{\sigma_0}$ is the map $\widetilde{\iota_1} : C(\mathbb{CP}^1) \to \widetilde{C_{\sigma_0}}$ sending $f \mapsto (f, f(\infty))$ and so $\widetilde{\iota_1}$ sends $H \in M_2(C(\mathbb{CP}^1))$ to

$$\left(\begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \overline{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \in M_2(\widetilde{C_{\sigma_0}}).$$

Meanwhile, ι_2 sends $E_{0,0} \in \mathbb{K}$ to $(0, E_{0,0}) \in C_{\sigma_0} \subseteq \widetilde{C_{\sigma_0}}$. Thus, $\widetilde{\iota}_{1*}([H]) - \iota_{2*}([E_{0,0}])$ is represented by the projection

$$P = \left(\begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \overline{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{bmatrix}, \begin{bmatrix} 1-E_{0,0} & 0 \\ 0 & 0 \end{bmatrix} \right) \in M_2(\widetilde{C_{\sigma_0}}).$$

Note this is precisely the range projection of the partial isometry

$$W = \left(\begin{bmatrix} z & 0 \\ \sqrt{1-|z|^2} & 0 \end{bmatrix}, \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \right) \in M_2(\widetilde{C_{\sigma_0}}).$$

whose source projection is

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

which proves that $\widetilde{\iota}_{1*}([H]) - \iota_{2*}([E_{0,0}]) = [1]$, as was needed. $\square$

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